UNCONDITIONAL BANACH SPACE IDEAL PROPERTY

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Abstract

Let $L^{w'}$ denote the assignment which associates with each pair of Banach spaces $X$, $Y$, the vector space $L^{w'}(X, Y)$ and $K(X, Y)$ be the space of all compact linear operators from $X$ to $Y$. Let $T \in L^{w'}(X, Y)$ and suppose $(T_n) \subset K(X, Y)$ converges in the dual weak operator topology ($w'$) of $T$. Denote by $K_u((T_n))$ the finite number given by

$$K_u((T_n)) := \sup_{n \in \mathbb{N}} \{ \max \{ \|T_n\|, \|T - 2T_n\| \} \}.$$ 

The $u$-norm on $L^{w'}(X, Y)$ is then given by

$$\|T\|_u := \inf \{ K_u((T_n)) : T = w' - \lim_n T_n, \ T_n \in K(X, Y) \}.$$ 

It has been shown that $(L^{w'}(X, Y), \|\cdot\|_u)$ is a Banach operator ideal. We find conditions for $K(X, Y)$ to be an unconditional ideal in $(L^{w'}(X, Y), \|\cdot\|_u)$.

1. Introduction

In Section 8 of paper [2], the authors established necessary conditions on a Banach space $X$ such that the space $K(X)$ of compact operators is a $u$-ideal in the space $L(X)$ of bounded linear operators, showing that this is the case if $X$ is separable and has (UKAP) (unconditional compact approximation property, i.e., if there exists a sequence $(K_n)$ in $K(X)$ such that $\lim_n K_n x = x$ for all $x \in X$ and $\lim_n \|id_X - 2K_n\| = 1$).

Johnson proved in [5] that if $Y$ is a Banach space having the bounded approximation property, then the annihilator $K(X, Y)^\perp$ in the (continuous) dual space $L(X, Y)^*$ is the kernel of a projection on $L(X, Y)^*$. The range space of the projection is isomorphic to the dual space $K(X, Y)^*$. John showed in [3] that Johnson’s result is also true in
case of any separable Pisier space \( X = P \) and its dual \( Y = P^* \), both
being spaces, which do not have the approximation property. This
motivated his more general results in a later paper (cf. [4]).

In the paper [1], an alternative (operator ideal) approach is followed
to prove similar (and more general) versions of John’s results. Having
proved that \((L^w, \left\| \cdot \right\|_u)\) is a Banach operator ideal (cf. [6]), we shall build
on the results in [1] to obtain conditions for the space \( K(X, Y) \) of compact
operators to be a \( u \)-ideal in a suitable subspace \((L^w(X, Y), \left\| \cdot \right\|_u)\) of
\( L(X, Y) \). If \((L^w(X, Y)) = L(X, Y)\), our results states conditions on
\( L(X, Y) \) so that \( K(X, Y) \) is a \( u \)-ideal in \( L(X, Y) \).

Before we investigate the \( u \)-ideal property of \( K(X, Y) \) in
\((L^w(X, Y), \left\| \cdot \right\|_u)\), we recall from [1], the ideal property of \( K(X, Y) \) in
\((L^w(X, Y)) \) with respect to the \( \| \cdot \| \)-norm.

**Theorem 1.1** (cf. [1], Theorem 2.5). There exists a continuous bilinear
form

\[ J : L^w(X, Y)^* \times L^w(X, Y) \rightarrow K, \]

such that

(a) \( J(\phi, T) = \phi(T) \) for all \( (\phi, T) \in L^w(X, Y)^* \times L^w(X, Y) \);

(b) \( |J(\phi, T)| \leq \|\phi\| \cdot \|T\| \) for all \( T \in L^w(X, Y) \) and \( \phi \in L^w(X, Y)^* \);

(c) \( J(\phi, T) = \lim_n \phi(T_n) \), where \( (T_n) \) is any sequence of compact
operators \( T_n \in K(X, Y) \) tending to \( T \) in \( w' \)-topology.

**Corollary 1.2.** Let \( X, Y \) be Banach spaces. There is a projection

\[ P : (L^w(X, Y), \left\| \cdot \right\|)^* \rightarrow (L^w(X, Y), \left\| \cdot \right\|)^*, \]
such that \( \text{Ker}(P) = K(X, Y)^{\perp} = \{ \phi \in L^{w'}(X, Y)^* : \phi \setminus K(X, Y) = 0 \} \), \( \|P\| \leq 1 \) and the range of \( P \) is isomorphic to \( K(X, Y)^* \). Thus \( K(X, Y) \) is an ideal in \( (L^{w'}(X, Y), \|\cdot\|) \). The projection \( P \) is given by
\[
P_{\phi}(T) = \lim_{n} \phi(T_n) = J(\phi, T),
\]
for all \( \phi \in L^{w'}(X, Y)^* \) and \( T \in L^{w'}(X, Y) \).

Since the norms \( \|\cdot\| \) and \( \|\cdot\| \) are equivalent when \( L(X, Y) = L^{w'}(X, Y) \), it follows from ([6], Corollary 2.7) that

**Corollary 1.3** (cf. [4]). Let \( X, Y \) be Banach spaces such that for each \( T \in L(X, Y) \), there is a sequence \( (T_n) \subset K(X, Y) \) such that \( w' T \). Then there exists a projection
\[
P : L(X, Y)^* \to L(X, Y)^*,
\]
such that
\[
\text{Ker}(P) = K(X, Y)^{\perp} = \{ \phi \in L^{w'}(X, Y)^* : \phi \setminus K(X, Y) = 0 \},
\]
and the range of \( P \) is isomorphic to \( K(X, Y)^* \).

2. Unconditional Ideal Property

The authors in [2] call a sequence \( (K_n) \) of compact operators from \( X \) into \( X \) a compact approximation sequence, if \( \lim_{n} K_n x = x \) for all \( x \in X \). In [2], it is also agreed to say that \( X \) has (UKAP), if there is a compact approximation sequence \( K_n : X \to X \) such that \( \lim_{n \to \infty} \|I - 2K_n\| = 1 \). It is also proved in [2] that if \( X \) is a separable Banach space, then \( X \) has (UKAP), if and only if for every \( \epsilon > 0 \), there is a sequence \( (A_n) \) of
compact operators such that for every $x \in X$ and every $n$ and every $0_j = \pm 1, 1 \leq j \leq n$, we have $\sum_{i=1}^{\infty} A_n x = x$ and $\| \sum_{i=1}^{\infty} 0_j A_i x \| \leq (1 + \epsilon) \| x \|$. In particular, this means that if we let $K_n = \sum_{i=1}^{\infty} A_i$, then $K_n x \to x$, $\forall x \in X$ and

$$\| K_n x \| \leq (1 + \epsilon) \| x \|, \quad \forall x \in X, \quad \forall n \in \mathbb{N}.$$ 

Moreover, also $\| I - 2K_n \| \leq 1 + \epsilon$.

When a separable Banach space $X$ has UKAP, it is easily seen that for each $T \in L(X), TK_n \to T$ (as $n \to \infty$) in the weak operator topology. If $X$ is also reflexive, then $TK_n \to T$ (as $n \to \infty$) in the $w'$-topology and it follows that

$$K_n ((TK_n)) \leq (1 + \epsilon) \| T \|.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\| T \|_{w'} \leq \| T \|$, i.e., $\| T \| = \| T \|_{w'}$ in this case. Putting $T_n := TK_n$, it follows that $T_n \to T$ and

$$\| T - 2T_n \| \leq (1 + \epsilon) \| T \|,$$

for all $n \in \mathbb{N}$. Consequently, it follows that

$$\| Id_{(L^w)^*} - 2P \| = \sup_{\| \phi \| \leq 1} \| \phi - 2P\phi \|$$

$$= \sup_{\| \phi \| \leq 1} \| \phi - 2P\phi \|$$

$$= \sup_{\| \phi \| \leq 1} \sup_{\| T \| \leq 1} | \phi(T) - 2P\phi(T) |$$

$$= \sup_{\| \phi \| \leq 1} \sup_{\| T \| \leq 1} \lim_{n} | \phi(T - 2T_n) |$$

$$\leq \sup_{\| T \| \leq 1} \sup_{n} \| T - 2T_n \| \leq 1 + \epsilon.$$

This being so for all $\epsilon > 0$, it is clear that
**Proposition 2.1** (Special case of [2], Proposition 8.2). *Let X be a separable reflexive Banach space. If X has (UKAP), then K(X) is a u-ideal in L(X).*

If X satisfies the conditions in Proposition 2.1 and Y is any Banach space, then for each $T \in L(X, Y)$ and each $\varepsilon > 0$, we may choose the sequence $(K_n) \subset K(X)$ to satisfy the properties in the above proof of Proposition 2.1. Again, put $T_n = TK_n$ for all $n$. Then, as before, $T_n \xrightarrow{w'} T$ and we still have the inequalities

$$
\|T - 2T_n\| \leq (1 + \varepsilon)\|T\| \quad \text{and} \quad K_u((T_n)) \leq (1 + \varepsilon)\|T\|.
$$

Hence $\|T\| \leq (1 + \varepsilon)\|T\|$ for all $\varepsilon > 0$. The existence of a contractive projection $P : L(X, Y)^* \to L(X, Y)^*$ with $\text{Ker}(P) = K(X, Y)^\perp$ follows from the Theorem 1.1 and Corollary 1.2, since in this case, we have $(L(X, Y), \|\|) = (L^{w'}(X, Y), \|\|_u)$. Therefore, $K(X, Y)$ is an ideal in $L(X, Y)$. The argument in the proof of Proposition 2.1, then shows that

**Proposition 2.2.** *Let X be a separable reflexive Banach space and Y be any Banach space. If X has (UKAP), then K(X, Y) is a u-ideal in L(X, Y).*

In the above discussion of the proof of Proposition 2.1, it is important to realize that for each $T \in L(X, Y)$ and each $\varepsilon > 0$, the sequence $(T_n) \subset K(X, Y)$ can be chosen to satisfy $T_n \xrightarrow{w'} T$ and $\|T - 2T_n\| \leq (1 + \varepsilon)\|T\|$ and $\|T_n\| \leq (1 + \varepsilon)\|T\|$. With the conditions on the Banach space X in Proposition 2.2, the norms $\|\|$, $\|\|_u$, and $\|\|_i$ coincide on $L(X, Y)$, exactly because we can choose the sequence $(T_n)$ as such. Therefore, it is natural to formulate the following definition:
**Definition 2.3.** Let $X$ and $Y$ be Banach spaces. We say an operator $T \in L(X, Y)$ has $(w' - \text{UKAP})$ (i.e., it has the “$w'$-uniform compact approximation property” if each $\epsilon > 0$, there exists a sequence $(T_n) \subset K(X, Y)$ such that $T = w' - \lim_n T_n$, $\|T - 2T_n\| \leq (1 + \epsilon)\|T\|$, and $\|T_n\| \leq (1 + \epsilon)\|T\|$ for all $n$. It follows from the above discussion that

**Proposition 2.4.** Suppose each $T \in (L^{w'}(X, Y), \|\cdot\|_w)$ (respectively, each $T \in L(X, Y)$) has $(w' - \text{UKAP})$. Then $K(X, Y)$ is a $u$-ideal in $(L^{w'}(X, Y), \|\cdot\|_w)$ (respectively, in $L(X, Y)$).

Although Proposition 2.1 is here discussed as a motivation for the condition $(w' - \text{UKAP})$ in Proposition 2.4, it was already proved in [2] (cf. Theorem 8.3) that a separable reflexive Banach space has (UKAP), if and only if $K(X)$ is a $u$-ideal in $L(X)$.

In this context, we may introduce yet another property on Banach spaces, as follows: A sequence $(K_n)$ of compact operators from $X$ into $X$ is called a $w'$-compact approximating sequence, if $w' - \lim_n K_n = I$. If $X$ is reflexive, then clearly each compact approximating sequence is $w'$-compact approximating. We say $X$ has $(w' - \text{UKAP})$ if for each $\epsilon > 0$, there is a $w'$-compact approximating sequence $K_n : X \to X$ such that $\|K_n x\| \leq (1 + \epsilon)\|x\|$, $\forall x \in X$, $\forall n \in \mathbb{N}$, and $\|I - 2K_n\| \leq 1 + \epsilon$ for all $n$. It then follows from Proposition 2.4 that

**Corollary 2.5.** If $X$ has $(w' - \text{UKAP})$, then $K(X)$ is a $u$-ideal in $L(X)$. 

References


